BA'ZI TRIGONOMETRIK AYNIYATLARNI VA TENGSIZLIKLARNI ISBOTLASH

Xondiyarov A. X.
Jizzax politexnika instituti talabasi

Rahimov B. Sh.
Jizzax politexnika instituti assistenti

Annotatsiya
Koʻpgina oʻquvchilar trigonometrik ayniyatlarni va tengsizliklarni isbotlashda qiynalishadi. Ushbu ishda ba'zi trigonometrik ayniyatlarni va tengsizliklarni isbotlarga doir misollar keltirib oʻtilgan.

Kalit soʻzlar: trigonometrik tenglama, trigonometrik tengsizlik, ayniyat, funksiya.

SOME TRIGONOMETRICAL EXERCISES AND INEQUALITIES PROVE

Toshnazarov B. U.
Student of Jizzakh Polytechnic Institute

Rahimov B. SH.
Assistant of Jizzakh Polytechnic Institute

Abstract
Many students find it difficult to prove trigonometric expressions and inequalities. In this work, examples of proving some trigonometric expressions and inequalities are given.

Keywords: Trigonometric equation, trigonometric inequality, identity, function.

1-misol. Agar $\alpha > 0,\ \beta > 0,\ \gamma > 0$ boʻlib, $\alpha + \beta + \gamma = \frac{\pi}{2}$ boʻlsa,
$tg\alpha \cdot tg\beta + tg\beta \cdot tg\gamma + tg\gamma \cdot tg\alpha = 1$ ayniyatni isbotlang.

Isbot. Ayniyatning chap qismidan oʻng qismini keltirib chiqaramiz:

$$tg\alpha \cdot tg\beta + tg\beta \cdot tg\gamma + tg\gamma \cdot tg\alpha = tg\alpha \cdot tg\beta + tg\gamma(tg\alpha + tg\beta) =$$

$$= tg\alpha \cdot tg\beta + tg\left(\frac{\pi}{2} - (\alpha + \beta)\right)(tg\alpha + tg\beta) = tg\alpha \cdot tg\beta + ctg(\alpha + \beta)(tg\alpha + tg\beta) =$$

$$= tg\alpha \cdot tg\beta + \frac{1 - tg\alpha \cdot tg\beta}{tg\alpha + tg\beta}(tg\alpha + tg\beta) = 1.$$

2-misol. Agar $-\arcsin\frac{2\sqrt{5}}{5} \le \alpha \le \pi - \arcsin\frac{2\sqrt{5}}{5}$ boʻlsa,
$\sqrt{4 - 3\sin^2\alpha + 2\sin 2\alpha} = \sin\alpha + 2\cos\alpha$ ayniyatni isbotlang.

Isbot. Ayniyatning chap qismidan

$$\sqrt{4-3\sin^2\alpha+2\sin 2\alpha}=\sqrt{4(\sin^2\alpha+\cos^2\alpha)-3\sin^2\alpha+4\sin\alpha\cos\alpha}=$$

$$=\sqrt{\sin^2\alpha+4\sin\alpha\cos\alpha+4\cos^2\alpha}=\sqrt{(\sin\alpha+2\cos\alpha)^2}=|\sin\alpha+2\cos\alpha|.$$

$\sin\alpha+2\cos\alpha$ ifodaning ishorasini aniqlash uchun quyidagi formuladan foydalanamiz:

$$a\cos\omega t+b\sin\omega t=\sqrt{a^2+b^2}\sin(\omega t+\varphi), \qquad (1)$$

bu yerda $\varphi=\arcsin\dfrac{a}{\sqrt{a^2+b^2}}$

(1) ga ko'ra $\sin\alpha+2\cos\alpha=\sqrt{5}\sin\left(\alpha+\arcsin\dfrac{2\sqrt{5}}{5}\right)$

Berilgan $-\arcsin\dfrac{2\sqrt{5}}{5}\le\alpha\le\pi-\arcsin\dfrac{2\sqrt{5}}{5}$ shartga ko'ra $0\le\alpha+\arcsin\dfrac{2\sqrt{5}}{5}\le\pi$ bo'ladi. Bunga ko'ra $\sin\left(\alpha+\arcsin\dfrac{2\sqrt{5}}{5}\right)\ge 0$, ya'ni $\sin\alpha+2\cos\alpha\ge 0$ ekanligi kelib chiqadi. Demak,

$|\sin\alpha+2\cos\alpha|=\sin\alpha+2\cos\alpha$ bo'lishi kelib chiqadi.

3-misol. Agar $tg\gamma=\dfrac{\sin\alpha\sin\beta}{\cos\alpha+\cos\beta}$ (2)

bo'lib, bunda

$$\begin{cases} 2\pi k<\gamma<\dfrac{\pi}{2}+2\pi k, \\ 2\pi m<\gamma<\dfrac{\pi}{2}+2\pi m, \\ 2\pi n<\gamma<\dfrac{\pi}{2}+2\pi n, \end{cases} \qquad (3)$$

bo'lsa,

$$tg\frac{\gamma}{2}=tg\frac{\alpha}{2}tg\frac{\beta}{2} \qquad (4)$$

bo'lishini isbotlang.

Isbot. (2) tenglikning ikkala qismini kvadratga ko'tarib, ikkala qismiga ham 1 ni qo'shamiz

$$1+tg^2\gamma=1+\frac{\sin^2\alpha\sin^2\beta}{(\cos\alpha+\cos\beta)^2}$$

va bundan

$$\frac{1}{\cos^2\gamma}=\frac{\cos^2\alpha+2\cos\alpha\cos\beta+\cos^2\beta+\sin^2\alpha\sin^2\beta}{(\cos\alpha+\cos\beta)^2},$$

$$\cos^2\gamma=\frac{(\cos\alpha+\cos\beta)^2}{\cos^2\alpha+2\cos\alpha\cos\beta+\cos^2\beta+(1-\cos^2\alpha)(1-\cos^2\beta)},$$

$$\cos^2\gamma = \frac{(\cos\alpha + \cos\beta)^2}{(1+\cos\alpha\cos\beta)^2},$$

yoki

$$|\cos\gamma| = \frac{|\cos\alpha + \cos\beta|}{|1+\cos\alpha\cos\beta|},$$

(3) shartlardan $\cos\gamma > 0, \quad \cos\alpha > 0, \quad \cos\beta > 0,$ ekanligi va bundan

$$\cos\gamma = \frac{\cos\alpha + \cos\beta}{1+\cos\alpha\cos\beta}, \qquad (4)$$

boʻlishi kelib chiqadi.

Proporsiyaning xossasiga koʻra

$$\frac{1-\cos\gamma}{1+\cos\gamma} = \frac{1+\cos\alpha\cos\beta - (\cos\alpha + \cos\beta)}{1+\cos\alpha\cos\beta + (\cos\alpha + \cos\beta)},$$

$$\frac{1-\cos\gamma}{1+\cos\gamma} = \frac{(1-\cos\beta) - \cos\alpha(1-\cos\beta)}{(1+\cos\beta) + \cos\alpha(1+\cos\beta)},$$

yoki

$$\frac{1-\cos\gamma}{1+\cos\gamma} = \frac{(1-\cos\beta)(1-\cos\alpha)}{(1+\cos\beta)(1+\cos\alpha)}, \qquad (5)$$

$1-\cos x = 2\sin^2\frac{x}{2}, \quad 1+\cos x = 2\cos^2\frac{x}{2},$ formulalarga koʻra (5) tenglikni quyidagicha yozish mumkin:

$$tg^2\frac{\gamma}{2} = tg^2\frac{\beta}{2}tg^2\frac{\alpha}{2}$$

bundan

$$\left|tg\frac{\gamma}{2}\right| = \left|tg\frac{\beta}{2}\right|\left|tg\frac{\alpha}{2}\right|$$

boʻlishi kelib chiqadi. (3) shartlarga koʻra $tg\frac{\gamma}{2} > 0, \quad tg\frac{\beta}{2} > 0, \quad tg\frac{\alpha}{2} > 0,$

Demak,

$$tg\frac{\gamma}{2} = tg\frac{\beta}{2}tg\frac{\alpha}{2}$$

tenglikga ega boʻlamiz.

4-misol. Agar $\alpha + \beta + \gamma = \pi$ boʻlsa, $\cos\alpha + \cos\beta + \cos\gamma \le \frac{3}{2}$ tengsizlikni isbotlang.

Isbot. Tengsizlikning chap qismidagi $\cos\alpha + \cos\beta$ yigʻindini koʻpaytmaga keltiramiz,

$$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + \cos\gamma$$

Bunda $\cos\frac{\alpha-\beta}{2} \le 1$ va $\frac{\alpha+\beta}{2} = \frac{\pi-\gamma}{2}$ ifodalarni e'tiborga olsak,

$$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\cos\gamma\le 2\cos\frac{\pi-\gamma}{2}+\cos\gamma=2\cos(\frac{\pi}{2}-\frac{\gamma}{2})+\cos\gamma=2\sin\frac{\gamma}{2}+\cos\gamma$$

$\cos\gamma$ ni yarim burchak formulalaridan foydalanib

$$2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\cos\gamma\le 2\sin\frac{\gamma}{2}+\cos\gamma=2\sin\frac{\gamma}{2}+1-2\sin^2\frac{\gamma}{2}$$

Bu yerda $\sin\frac{\gamma}{2}=y$ belgilash kiritsak, $-2y^2+2y+1$ funksiyaga ega bo‘lamiz. Bu kvadratik funksiyaning eng katta qiymati $\frac{3}{2}$ ga teng. Bundan esa isbotlanishi talab etilgan tengsizlikka ega bo‘lamiz.

5-misol. $\sin^3\alpha(1+ctg\alpha)+\cos^3\alpha(1+tg\alpha)<\frac{m^4+1}{m^2}$ tengsizlikni isbotlang.

Isbot. Biz bilamizki $m^2+\frac{1}{m^2}>2$ tengsizlik o‘rinli. Demak, biz

$$\sin^3\alpha(1+ctg\alpha)+\cos^3\alpha(1+tg\alpha)<2$$

tengsizlikni isbotlashimiz kerak.

$$\sin^3\alpha(1+ctg\alpha)+\cos^3\alpha(1+tg\alpha)=\sin^3\alpha\frac{\sin\alpha+\cos\alpha}{\sin\alpha}+\cos^3\alpha\frac{\sin\alpha+\cos\alpha}{\cos\alpha}=$$
$$=\sin^2\alpha(\sin\alpha+\cos\alpha)+\cos^2\alpha(\sin\alpha+\cos\alpha)=(\sin\alpha+\cos\alpha)(\sin^2\alpha+\cos^2\alpha)=$$
$$=\sin\alpha+\cos\alpha=\sqrt{2}\cos\left(\alpha-\frac{\pi}{4}\right)<2$$

Tengsizlik isbot bo‘ldi.

6-misol. Ayniyatni isbotlang.

$$\frac{1}{\cos x\cdot\cos 2x}+\frac{1}{\cos 2x\cdot\cos 3x}+\dots+\frac{1}{\cos 9x\cdot\cos 10x}=\frac{2\sin 9x}{\sin 2x\cdot\cos 10x}$$

Isbot. Tengsizlikni isbotlash uchun $\sin x=\sin(nx-(n-1)x)$ ifodadan foydalanamiz, ya’ni berilgan ifodaning chap qismiga *sinx* ni ko‘paytirib bo‘lamiz,

$$\frac{1}{\sin x}\left(\frac{\sin x}{\cos x\cdot\cos 2x}+\frac{\sin x}{\cos 2x\cdot\cos 3x}+\dots+\frac{\sin x}{\cos 9x\cdot\cos 10x}\right)=$$
$$=\frac{1}{\sin x}\left(\frac{\sin(2x-x)}{\cos x\cdot\cos 2x}+\frac{\sin(3x-2x)}{\cos 2x\cdot\cos 3x}+\dots+\frac{\sin(10x-9x)}{\cos 9x\cdot\cos 10x}\right)=$$

qo‘shish formulalariga ko‘ra

$$= \frac{1}{\sin x}\left(\frac{\sin 2x\cos x - \cos 2x\sin x}{\cos x \cdot \cos 2x} + \frac{\sin 3x\cos 2x - \sin 2x\cos 3x}{\cos 2x \cdot \cos 3x} + \cdots + \right.$$

$$\left. + \frac{\sin 10x\cos 9x - \sin 9x\cos 10x}{\cos 9x \cdot \cos 10x} \right) = \frac{1}{\sin x}\left(tg2x - tgx + tg3x - tg2x + \cdots + tg10x - tg9x\right) =$$

$$= \frac{1}{\sin x}\left(tg10x - tgx\right) = \frac{1}{\sin x}\left(\frac{\sin 10x}{\cos 10x} - \frac{\sin x}{\cos x} \right) = \frac{1}{\sin x}\frac{\sin 10x\cos x - \sin x\cos 10x}{\cos 10x\cos x} =$$

$$= \frac{2\sin 9x}{\sin 2x\cos 10x}$$

Ayniyat isbotlandi.

Foydalanilgan adabiyotlar:

1. В.Н.Литвененко, А.Г.Мордкович Практикум по решению задач школьной математики Москва “Просвещение” 1977
2. Останов К., Тураев У. Я., Рахимов Б. Ш. Об обучении учащихся основным методам решения квадратных неравенств //European science. – 2020. – №. 1 (50). – С. 57-59.
3. Ne’Matov A. R. et al. Aniq integralni me’morchilikda qo‘llash. Aniq integralning tadbiqlariga doir misollar yechish //Science and Education. – 2022. – Т. 3. – №. 2. – С. 16-21.
4. Rahimov B. S. Matematik tushunchalarni kiritish va tavsiflash usullari //Science and Education. – 2022. – Т. 3. – №. 5. – С. 951-956.
5. Тураев У. Я. и др. Ценность матричной игры принцип минимакса и его экономический анализ //Science and Education. – 2022. – Т. 3. – №. 5. – С. 126-136.
6. Shermuxammadovich R. B., Qaxramon A. OLIY TA’LIM MUASSASALARIDA INNOVATSIYALAR MASALASI HAQIDA //Uzbek Scholar Journal. – 2024. – Т. 27. – С. 1-4.
7. Azimov Q., Sh R. B. RISK SHAROITIDA YECHIM QABUL QILISH //Экономика и социум. – 2024. – №. 2 (117)-1. – С. 113-116.
8. Turaev U., Rakhimov B. THE VALUE OF THE MATRIX GAME, THE MINIMAX PRINCIPLE AND ITS ECONOMIC ANALYSIS //Science and Innovation. – 2022. – Т. 1. – №. 3. – С. 36-40.
9. Rahimov B. S. H., Ne’matov A. R., Fayzullayev S. E. Lagranj funksiyasidan foydalanib ba’zi masalalarni yechish haqida //Archive of Conferences. – 2022. – С. 41-43.
10. Fayzullayev S. et al. SHORT TERM LIFE INSURANCE MODELS //Science and Innovation. – 2022. – Т. 1. – №. 3. – С. 146-149.
11.Ne’matov A. Turli matnli masalalarni yechish usullari //Евразийский журнал математической теории и компьютерных наук. – 2023. – Т. 3. – №. 3. – С. 7-10.
12. Abdukadirovich S. U., Abduganievich D. U. ON SOME PROBLEMS OF EXTREME PROPERTIES OF THE FUNCTION AND THE APPLICATION OF THE DERIVATIVE AND METHODS FOR THEIR SOLUTION //Archive of Conferences. – 2021. – С. 113-117.